2.6 RATIONAL FUNCTIONS

What You Should Learn

- Find the domains of rational functions.
- Find the vertical and horizontal asymptotes of graphs of rational functions.
- Analyze and sketch graphs of rational functions.
- Sketch graphs of rational functions that have slant asymptotes.
- Use rational functions to model and solve real-life problems.

Introduction

Introduction

A **rational function** is a quotient of polynomial functions. It can be written in the form

$$f(x) = \frac{N(x)}{D(x)}$$

where $N(x)$ and $D(x)$ are polynomials and $D(x)$ is not the zero polynomial.

Example 1 – *Finding the Domain of a Rational Function*

Find the domain of the reciprocal function $f(x) = \frac{1}{x}$ and discuss the behavior of *f* near any excluded *x*-values.

Solution:

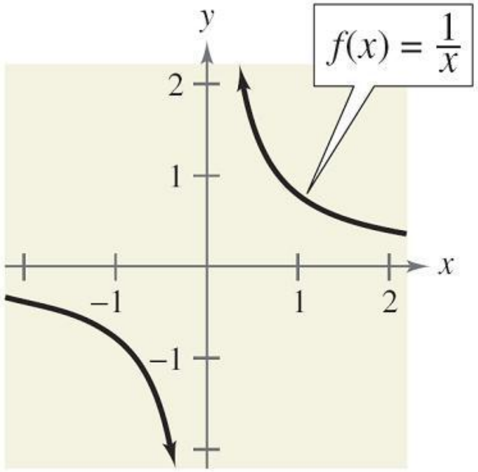
Because the denominator is zero when $x = 0$ the domain of *f* is all real numbers except $x = 0$.

Example 1 – *Solution*

cont'd

x	-1	-0.5	-0.1	-0.01	-0.001	$\rightarrow 0$
$f(x)$	-1	-2	-10	-100	-1000	$\rightarrow -\infty$

x	$0 \leftarrow$	0.001	0.01	0.1	0.5	1
$f(x)$	$\infty \leftarrow$	1000	100	10	2	1



Vertical and Horizontal Asymptotes

Vertical and Horizontal Asymptotes

$f(x) \longrightarrow -\infty$ as $x \longrightarrow 0^-$

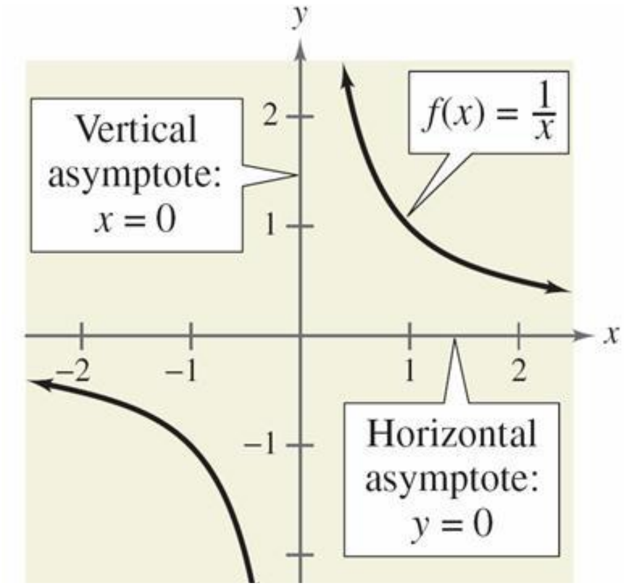
$f(x)$ decreases without bound as x approaches 0 from the left.

$f(x) \longrightarrow \infty$ as $x \longrightarrow 0^+$

$f(x)$ increases without bound as x approaches 0 from the right.

The line $x = 0$ is a **vertical asymptote** of the graph of f.

The line $y = 0$ is a **horizontal asymptote** of the graph of f.



$f(x) \longrightarrow 0$ as $x \longrightarrow -\infty$

$f(x)$ approaches 0 as x decreases without bound.

$f(x) \longrightarrow 0$ as $x \longrightarrow \infty$

$f(x)$ approaches 0 as x increases without bound.

Vertical and Horizontal Asymptotes

Definitions of Vertical and Horizontal Asymptotes

1. The line $x = a$ is a **vertical asymptote** of the graph of f if

$$f(x) \longrightarrow \infty \quad \text{or} \quad f(x) \longrightarrow -\infty$$

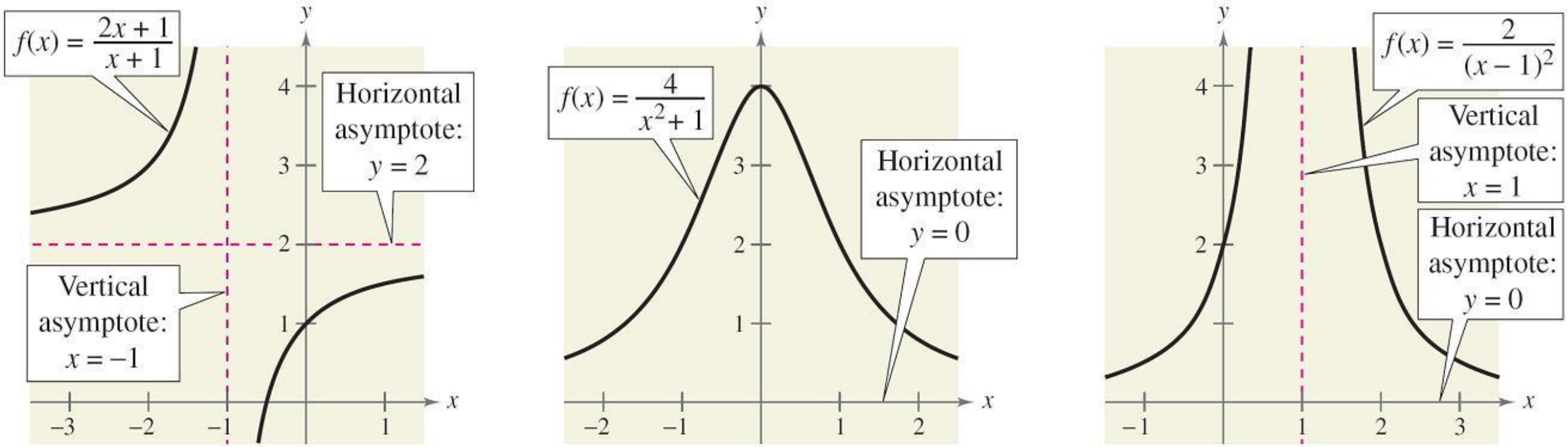
as $x \longrightarrow a$, either from the right or from the left.

2. The line $y = b$ is a **horizontal asymptote** of the graph of f if

$$f(x) \longrightarrow b$$

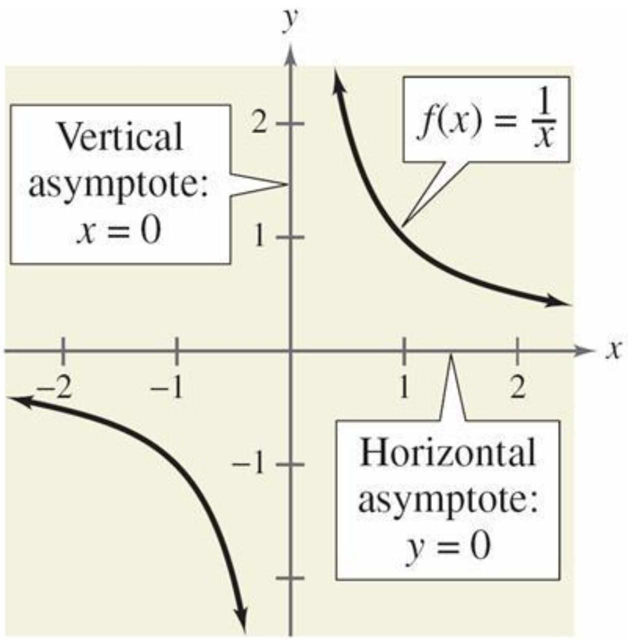
as $x \longrightarrow \infty$ or $x \longrightarrow -\infty$.

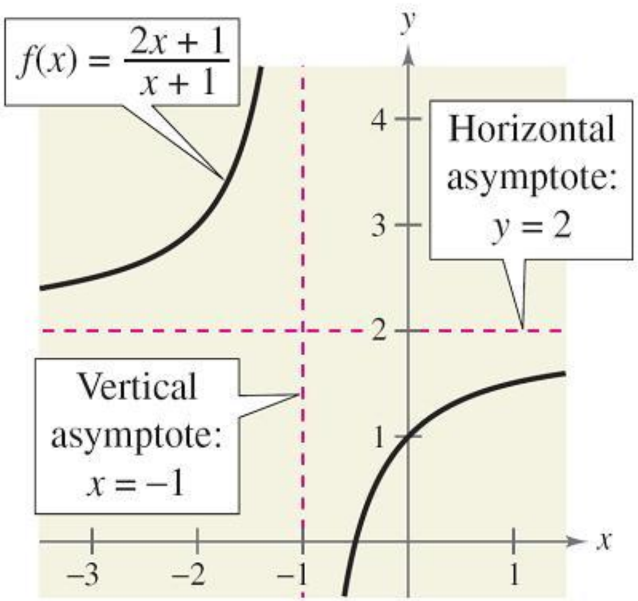
Vertical and Horizontal Asymptotes



Vertical and Horizontal Asymptotes

The graphs of $f(x) = \dfrac{1}{x}$ and $f(x) = \dfrac{2x + 1}{x + 1}$ are **hyperbolas.**





Vertical and Horizontal Asymptotes

Vertical and Horizontal Asymptotes of a Rational Function

Let f be the rational function given by

$$f(x) = \frac{N(x)}{D(x)} = \frac{a_n x^n + a_{n-1} x^{n-1} + \cdot \cdot \cdot + a_1 x + a_0}{b_m x^m + b_{m-1} x^{m-1} + \cdot \cdot \cdot + b_1 x + b_0}$$

where $N(x)$ and $D(x)$ have no common factors.

1. The graph of f has *vertical* asymptotes at the zeros of $D(x)$.

2. The graph of f has one or no *horizontal* asymptote determined by comparing the degrees of $N(x)$ and $D(x)$.

a. If $n < m$, the graph of f has the line $y = 0$ (the x-axis) as a horizontal asymptote.

b. If $n = m$, the graph of f has the line $y = \frac{a_n}{b_m}$ (ratio of the leading coefficients) as a horizontal asymptote.

c. If $n > m$, the graph of f has no horizontal asymptote.

Example 2 – *Finding Vertical and Horizontal Asymptotes*

Find all vertical and horizontal asymptotes of the graph of each rational function.

a. $f(x) = \dfrac{2x^2}{x^2 - 1}$ **b.** $f(x) = \dfrac{x^2 + x - 2}{x^2 - x - 6}$

Solution:

a. the degree of the numerator = the degree of the denominator.

The leading coefficient of the numerator is 2 and the leading coefficient of the denominator is 1, so the graph has the line $y = \frac{2}{1} = 2$ as a horizontal asymptote.

Example 2 – *Solution*

cont'd

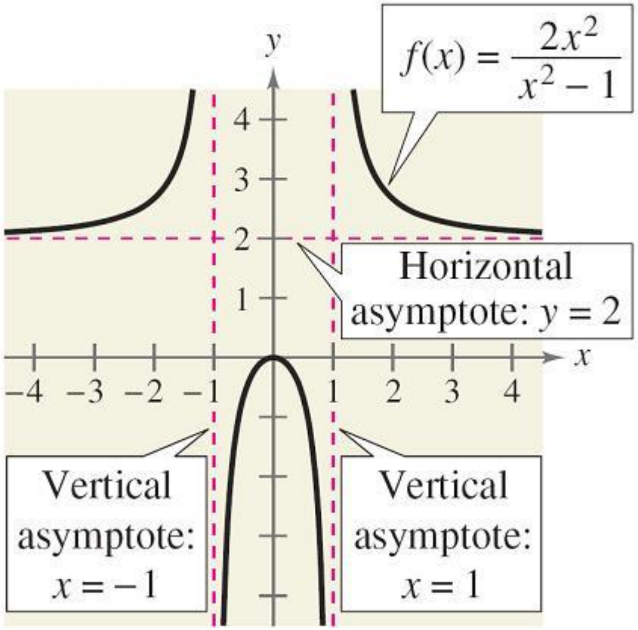
Denominator = 0

$x^2 - 1 = 0$	Set denominator equal to zero.
$(x + 1)(x - 1) = 0$	Factor.
$x + 1 = 0 \Rightarrow x = -1$	Set 1st factor equal to 0.
$x - 1 = 0 \Rightarrow x = 1$	Set 2nd factor equal to 0.

Example 2 – *Solution*

cont'd

The graph has the lines x = –1 and x = 1 as vertical asymptotes.



Example 2 – *Solution*

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b. $f(x) = \dfrac{x^2 + x - 2}{x^2 - x - 6}$

the degree of the numerator = the degree of the denominator

Horizontal asymptote:

$$y = \frac{1}{1} = 1$$

Vertical asymptotes:

$$f(x) = \frac{x^2 + x - 2}{x^2 - x - 6} = \frac{(x - 1)\cancel{(x + 2)}}{\cancel{(x + 2)}(x - 3)} = \frac{x - 1}{x - 3}, \quad x \neq -2$$

$$x = 3$$

Analyzing Graphs of Rational Functions

Analyzing Graphs of Rational Functions

Guidelines for Analyzing Graphs of Rational Functions

Let $f(x) = \dfrac{N(x)}{D(x)}$, where $N(x)$ and $D(x)$ are polynomials.

1. Simplify f, if possible.

2. Find and plot the y-intercept (if any) by evaluating $f(0)$.

3. Find the zeros of the numerator (if any) by solving the equation $N(x) = 0$. Then plot the corresponding x-intercepts.

4. Find the zeros of the denominator (if any) by solving the equation $D(x) = 0$. Then sketch the corresponding vertical asymptotes.

5. Find and sketch the horizontal asymptote (if any) by using the rule for finding the horizontal asymptote of a rational function.

6. Plot at least one point *between* and one point *beyond* each x-intercept and vertical asymptote.

7. Use smooth curves to complete the graph between and beyond the vertical asymptotes.

Example 3 – *Sketching the Graph of a Rational Function*

Sketch the graph of $g(x) = \dfrac{3}{x - 2}$ and state its domain.

Solution:

y-intercept: $\left(0, -\frac{3}{2}\right)$, because $g(0) = -\frac{3}{2}$

x-intercept: None, because $3 \neq 0$

Vertical asymptote: $x = 2$, zero of denominator

Horizontal asymptote: $y = 0$ because degree of $N(x) <$ degree of $D(x)$

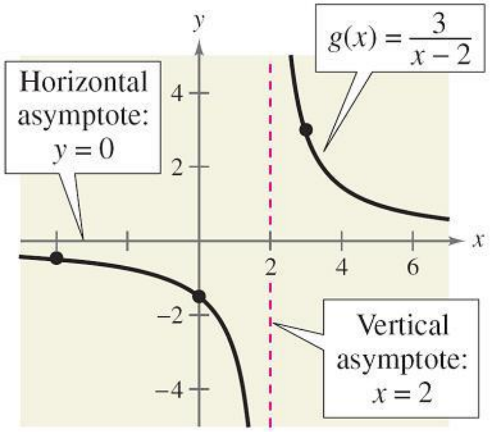
Example 3 – *Solution*

cont'd

Additional points:

Test interval	Representative x-value	Value of g	Sign	Point on graph
$(-\infty, 2)$	-4	$g(-4) = -0.5$	Negative	$(-4, -0.5)$
$(2, \infty)$	3	$g(3) = 3$	Positive	$(3, 3)$

The domain of g is all real numbers x except $x = 2$.



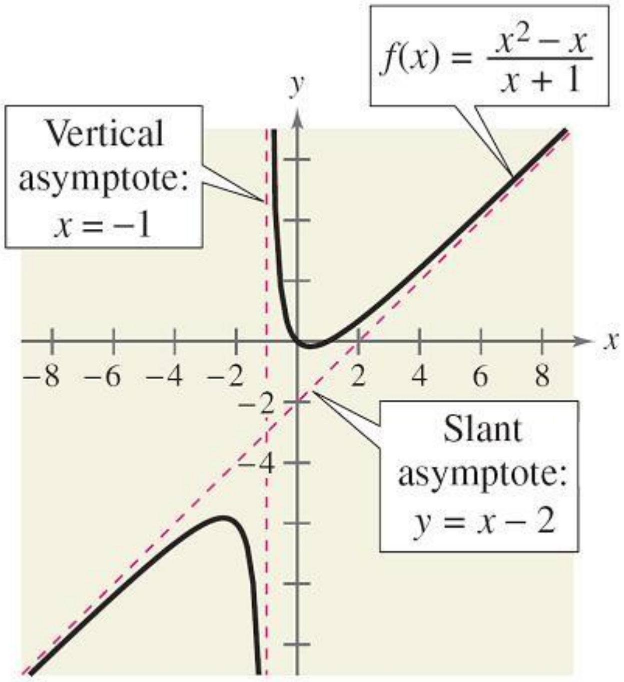
Slant Asymptotes

Slant Asymptotes

If the degree of the numerator is exactly *one more* than the degree of the denominator, the graph of the function has a **slant** (or **oblique**) **asymptote.**

the graph of $f(x) = \dfrac{x^2 - x}{x + 1}$

has a slant asymptote

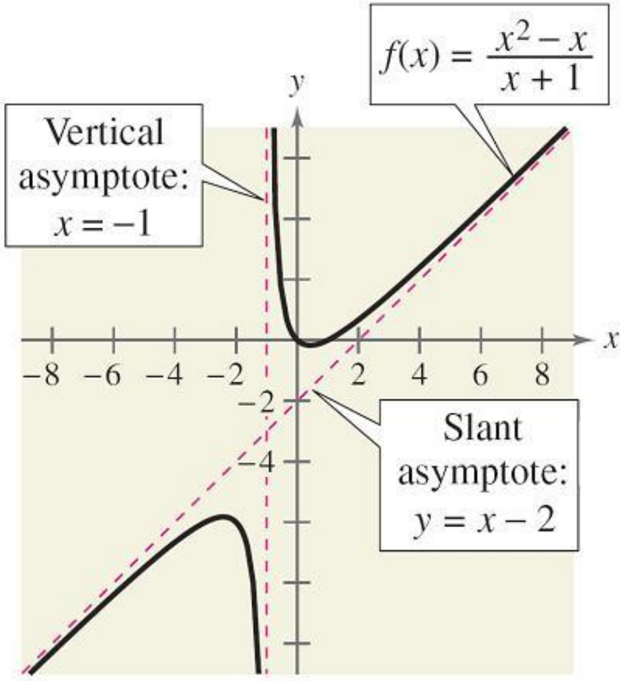


Slant Asymptotes

To find the equation of a slant asymptote, use long division.

$$f(x) = \frac{x^2 - x}{x + 1} = \underbrace{x - 2}_{\text{Slant asymptote } (y = x - 2)} + \frac{2}{x + 1}.$$

As x increases or decreases without bound, the remainder term $2/(x + 1)$ approaches 0, so the graph of f approaches the line $y = x - 2$.



Example 7 – *A Rational Function with a Slant Asymptote*

Sketch the graph of $f(x) = \dfrac{x^2 - x - 2}{x - 1}$.

Solution:

Factoring the numerator as (x – 2)(x + 1) allows you to recognize the x-intercepts.

Using long division

$$f(x) = \frac{x^2 - x - 2}{x - 1} = x - \frac{2}{x - 1}$$

allows you to recognize that the line $y = x$ is a slant asymptote of the graph.

Example 7 – *Solution*

cont'd

y-intercept: (0, 2), because $f(0) = 2$

x-intercepts: (–1, 0) and (2, 0)

Vertical asymptote: $x = 1$, zero of denominator

Slant asymptote: $y = x$

Additional points:

Test interval	Representative x-value	Value of f	Sign	Point on graph
$(-\infty, -1)$	-2	$f(-2) = -1.33$	Negative	$(-2, -1.33)$
$(-1, 1)$	0.5	$f(0.5) = 4.5$	Positive	$(0.5, 4.5)$
$(1, 2)$	1.5	$f(1.5) = -2.5$	Negative	$(1.5, -2.5)$
$(2, \infty)$	3	$f(3) = 2$	Positive	$(3, 2)$

Example 7 – *Solution*

cont'd

The graph is shown in Figure 2.46.

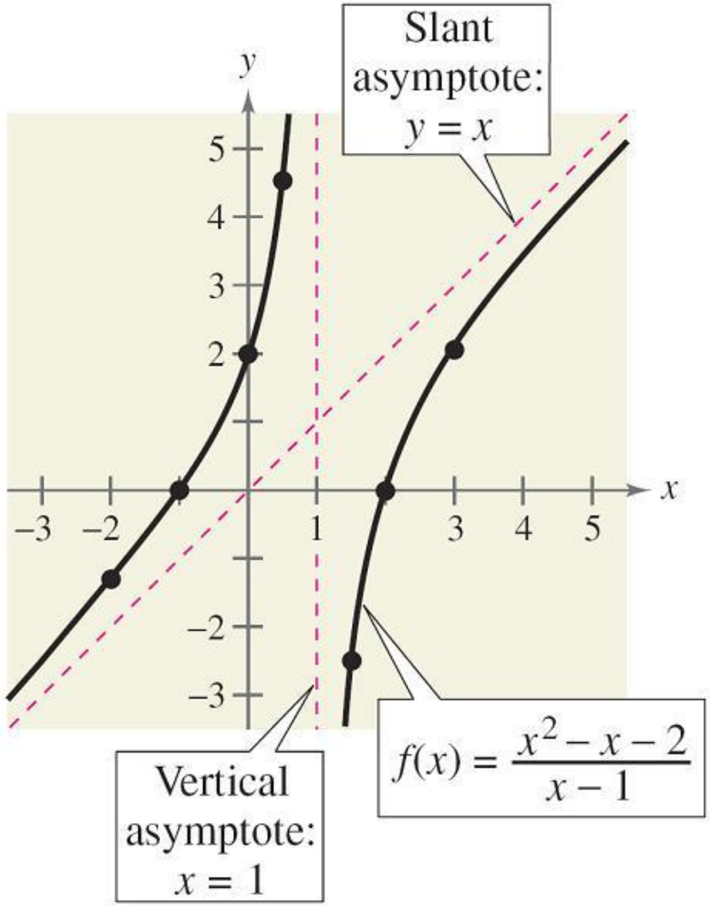


Figure 2.46

Applications

Example 8 – *Cost-Benefit Model*

A utility company burns coal to generate electricity. The cost C (in dollars) of removing p% of the smokestack pollutants is given by

$$C = \frac{80{,}000p}{100 - p}$$

for $0 \leq p < 100$. You are a member of a state legislature considering a law that would require utility companies to remove 90% of the pollutants from their smokestack emissions. The current law requires 85% removal. How much additional cost would the utility company incur as a result of the new law?

Example 8 – *Solution*

Because the current law requires 85% removal, the current cost to the utility company is

$$C = \frac{80{,}000(85)}{100 - 85} \approx \$453{,}333.$$ Evaluate C when $p = 85$.

If the new law increases the percent removal to 90%, the cost will be

$$C = \frac{80{,}000(90)}{100 - 90} = \$720{,}000.$$ Evaluate C when $p = 90$.

Example 8 – *Solution*

cont'd

So, the new law would require the utility company to spend an additional

720,000 – 453,333 = $266,667.

Subtract 85% removal cost from 90% removal cost.