11.1 Sequences and Series
Definition of a Sequence

An Arithmetic Sequence is a function whose domain is the set of positive integers.

The function values a_1, a_2, a_3, a_4... a_n are the terms of the sequence. (Sometimes will start a_0)

If the domain of the function consists of the first n positive integers only, the sequence is Finite

When you are given a definition for a_n you can find the different terms of a sequence.
Examples:

a) The first four terms of the sequence given by $a_n = 3n - 2$ are

$a_1 = 3(1) - 2 = 1$ 1st term

$a_2 = 3(2) - 2 = 4$ 2nd term

$a_3 = 3(3) - 2 = 7$ 3rd term

$a_4 = 3(4) - 2 = 10$ 4th term

b) The first four terms of the sequence given by $a_n = 3 + (-1)^n$ are

$a_1 = 3 + (-1)^1 = 2$ 1st term

$a_2 = 3 + (-1)^2 = 4$ 2nd term

$a_3 =$ 2 3rd term

$a_4 =$ 4 4th term

c) The first five terms of the sequence given by $a_n = \frac{(-1)^n}{2n-1}$ are

$a_1 = -1$ 1st term

$a_2 = 1/3$ 2nd term

$a_3 = -1/5$ 3rd term

$a_4 = 1/7$ 4th term

$a_5 = -1/9$ 5th term

Write out the first five terms of the sequence whose nth term is $a_n = \frac{(-1)^{n+1}}{2n-1}$.

$a_1 = 1$

$a_2 = -1/3$

$a_3 = 1/5$

$a_4 = -1/7$

$a_5 = 1/9$

Are they the same as the first five terms of the sequence in Example c? If not, how do they differ?

The signs are different because the n-term formula changed.

Now let's try to figure out the function used to find the nth term given the first few terms of the sequence. **Sometimes the beginnings of difference sequences will look the same, it is important to look at ALL of the terms that are given in order to define the sequence.**

For example:

$$\frac{1}{2},\frac{1}{4},\frac{1}{8},\frac{1}{16},\ldots,\frac{1}{2^n},\ldots \qquad \frac{1}{2},\frac{1}{4},\frac{1}{8},\frac{1}{15},\ldots,\frac{6}{(n+1)(n^2-n+6)},\ldots$$

These start off the same way, but end up being VERY DIFFERENT sequences! All we can do when we are figuring out what the nth term might be is look for the "apparent" pattern (aka – What it looks like it might be doing) because we won't have every term of the sequence to check it with.

Write an expression for the apparent nth term of each sequence

a) 1, 3, 5, 7, …
(First step is to list the term number with its term and see how the two numbers are related)

n:	1	2	3	4	…	n
Terms:	1	3	5	7	…	a_n

Apparent Pattern: Each term is 1 less than twice n, which implies that $a_n = 2n-1$

b) 2, 5, 10, 17, …

n:	1	2	3	4	…	n
Terms:	2	5	10	17		

Apparent Pattern: n^2+1

c) $2,\frac{3}{2},\frac{4}{3},\frac{5}{4},\ldots$ $\frac{n+1}{n}$

Our goal with these three have been to define a_n by n so that each term can be found independently from the other terms of the sequence. This is not always possible.

Recursive **Sequences** : Once you are given the first few terms, all of the other terms are defined using previous terms.

FAMOUS EXAMPLE: The Fibonacci Sequence: A Recursive Sequence

Definition: $a_0=1,\ a_1=1,\ a_k=a_{k-2}+a_{k-1}$ *In English –To find the next term, add the previous 2 terms

Write the first six terms:

$a_0=1$

$a_1=1$

$a_2=a_{2-2}+a_{2-1}=a_0+a_1=1+1=2$

$a_3=a_{3-2}+a_{3-1}=$ 3

$a_4=$ 5

$a_5=$ 8

Factorial - If n is a positive integer, *n factorial* is defined by $n!=1\cdot2\cdot3\cdot4\cdot\ldots\cdot(n-1)\cdot n$

Special Case~ $0!=1$.

Here are some values of n!

$0! = 1$

$1! = 1$

$2! = 1 \cdot 2 = 2$

$3! = 1 \cdot 2 \cdot 3 = 6$

$4! = 1 \cdot 2 \cdot 3 \cdot 4 = 24$

$5! = 1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 = 120$

BEWARE of your order of operations with factorials

$2n! \neq (2n)!$

$2n! = 2(1 \cdot 2 \cdot 3 \cdot \ldots \cdot n)$

$(2n)! = 1 \cdot 2 \cdot 3 \cdot \ldots \cdot 2n$

Example with Sequences:

List the first five terms of the sequence given by $a_n = \dfrac{2^n}{n!}$

$a_0 = \dfrac{2^0}{0!} = \dfrac{1}{1} = 1$

$a_1 = \dfrac{2^1}{1!} = 2$

$a_2 = \dfrac{4}{2!} = 2$

$a_3 = \dfrac{4}{3}$

$a_4 = \dfrac{2}{3}$

As n gets bigger, factorials get harder to do in your head!

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Factorial Expressions

Evaluate the expression without your calculator. Work "smart"

$$\frac{8!}{2! \cdot 6!}$$

$$\frac{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot 7 \cdot 8}{1 \cdot 2 \cdot 1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6}$$

Here's the "smart" technique $\dfrac{(1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6) \cdot 7 \cdot 8}{1 \cdot 2 \cdot (1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6)} = \dfrac{6! \cdot 7 \cdot 8}{1 \cdot 2 \cdot 6!} = \dfrac{7 \cdot 8}{1 \cdot 2} = \dfrac{56}{2} = 28$

Try these: $\dfrac{2! \cdot 6!}{3! \cdot 5!}$ $\dfrac{n!}{(n-1)!}$

$$\frac{\cancel{2} \cdot 6 \cdot \cancel{5} \cdot \cancel{4} \cdot \cancel{3} \cdot \cancel{2}}{3 \cdot \cancel{2} \cdot \cancel{5} \cdot \cancel{4} \cdot \cancel{3} \cdot \cancel{2}}$$

2

$$\frac{n\,\cancel{(n-1)}\,\cancel{(n-2)} \cdots \cancel{1}}{\cancel{(n-1)}\,\cancel{(n-2)} \cdots \cancel{1}}$$

n

Summation Notation / Sigma Notation

The sum of the first n terms of a sequence is represented by $\sum_{i=1}^{n} a_i = a_1 + a_2 + a_3 + \ldots + a_n$

Where i is called the index of summation, n is the upper limit of summation, and 1 is the lower limit of summation.

Example

a) $\sum_{i=1}^{5} 3i$

45

b) $\sum_{k=3}^{6} (1+k^2)$

90

c) $\sum_{i=0}^{8} \frac{1}{i!}$

2.718

Properties of Sums

1. $\sum_{i=1}^{n} ca_i = c\sum_{i=1}^{n} a_i$ c is any constant
2. $\sum_{i=1}^{n} (a_i + b_i) = \sum_{i=1}^{n} a_i + \sum_{i=1}^{n} b_i$
3. $\sum_{i=1}^{n} (a_i - b_i) = \sum_{i=1}^{n} a_i - \sum_{i=1}^{n} b_i$

Series

Consider the infinite sequence $a_1, a_2, a_3, \ldots a_i, \ldots$

1. The sum of all terms of the infinite sequence is called an **infinite series** and is denoted by

   $a_1 + a_2 + a_3 + \ldots + a_i + \ldots = \sum_{i=1}^{\infty} a_i$

2. The sum of the first n terms of the sequence is called a **finite series** or the ***n*th partial sum** of the sequence and is denoted by

   $a_1 + a_2 + a_3 + \ldots + a_n = \sum_{i=1}^{n} a_i$

Example

$\sum_{i=1}^{\infty} \frac{3}{10^i}$

a) Find the third partial sum 0.333

b) Find the sum 0.333... $= \frac{1}{3}$

Applications

For the years 1960-1997, the resident population of the United States can be approximated by the model

$$a_n = \sqrt{33{,}282 + 801.3n + 6.12n^2} \quad n = 0, 1, \ldots, 37$$

where a_n is the population in millions and n represents the calendar year, with $n = 0$ corresponding to 1960. Find the last five terms of this finite sequence, which represent the U.S. population for the years 1993 to 1997. 257.7, 260.0, 262.3, 264.7, 267.0